The M-matrix inverse problem for the Sturm-Liouville equation on graphs *

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Abstract

We consider an inverse spectral problem for Sturm-Liouville boundary value problems on a graph with formally self-adjoint boundary conditions at the nodes, where the given information is the M-matrix. Based on the results found in S. Currie, B.A. Watson, M-matrix asymptotics for Sturm-Liouville problems on graphs, J. Com. Appl. Math., doi: 10.1016/j.cam.2007.11.019, using the Green’s function, we prove that the poles of the M-matrix are at the eigenvalues of the associated boundary value problem and are simple, located on the real axis and that the residue at a pole is a negative semi-definite matrix with rank equal to the multiplicity of the eigenvalue. We define the so called norming constants and relate them to the spectral measure and the M-matrix. This enables us to recover, from the M-matrix, the boundary conditions and the potential, up to a unitary equivalence for co-normal boundary conditions.

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1 Introduction

In this paper we consider the second order differential equation

\[ ly := -\frac{d^2 y}{dx^2} + q(x)y = \lambda y, \]

(1.1)

where \( q \) is real-valued and continuous, on the weighted graph \( G \) with boundary conditions at the nodes formally self-adjoint with respect to \( l \) in \( L^2(G) \). For a characterisation of self-adjoint boundary value problems on graphs and formally self-adjoint boundary conditions, see [5] and [18].

Differential operators on graphs often appear in mathematics, mechanics, physics, geophysics, chemistry and engineering, see [12, 13, 20, 21, 24, 25] and the bibliographies thereof. In recent years the interest in the spectral theory of Sturm-Liouville equations on graphs has grown considerably although most of the research in this area is devoted to the so-called direct problem of studying the properties of the spectrum, see, for example [1, 29, 30]. At present there is no general theory for the spectral inverse problem for Sturm-Liouville operators on graphs. For some recent progress in the area see, [27, 28]. There have however been substantial advances on inverse spectral problems for Sturm-Liouville operators on trees, a special case of that on graphs, see [15] and [31]. In particular Brown and Weikard solve the inverse problem considered here on trees, see [4].

Various inverse spectral problems involving the recovery of the potential and boundary conditions (not dependent on the eigenparameter) for a scalar Sturm-Liouville problem were considered by Gel’fand and Levitan, Hochstadt, Krein and Marčenko in [16, 19, 22, 23, 26]. We also note that recently Bennewitz, [2], gave a very elegant solution of the m-function inverse problem. Amongst others, Binding, Browne and Watson and Chugnova, in [3, 7], consider inverse spectral problems for scalar Sturm-Liouville problems where the boundary conditions are dependent on the eigenparameter.

This paper is a continuation of [11], where we considered an M-matrix associated with a system formulation of the Sturm-Liouville operator, with formally self-adjoint boundary conditions, on a graph. There, the M-matrix was related to the matrix Prüfer angle of the system boundary value problem, and, consequently, with the boundary value problem on the graph. Asymptotics for the M-matrix were obtained as the eigenparameter tended to negative infinity. It was shown that the M-matrix is a matrix Herglotz function and the boundary conditions were recovered, from the M-matrix, up to a unitary equivalence. Here we show, for co-normal boundary conditions, see appendix, that the potential can be uniquely recovered, up to a unitary equivalence, from the M-matrix, see Theorem 5.3 and Corollary 5.4. In particular instances the problem is uniquely determined, see Corollary 5.5.

In section 2, preliminaries from [11] are recalled, while, in section 3, after proving various matrix Wronskian identities and obtaining a Green’s function for the problem, we show
that the poles of the M-matrix are simple, located at the eigenvalues of the boundary value problem and are on the real axis. It is also shown that the residues at the poles of the M-matrix are negative semi-definite matrices of rank equal to the multiplicity of the eigenvalue, see Theorem 3.3. Norming constants and the spectral measure are considered in section 4. In particular we relate the spectral measure to the M-matrix. The inverse problem is then solved in section 5. We build on the work of Marčenko, [26], (for scalar problems) to define a transformation operator which is also a Volterra operator. As a consequence, the potential can be recovered, from the M-matrix, up to a unitary equivalence for co-normal boundary conditions.

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2 Preliminaries

We consider (1.1) on the graph $G$ with edges $e_i, i = 1, \ldots, K$ with length $l_i$ respectively. From [9], equation (1.1) can be rewritten as

$$-y''_i(x) + q_i(x)y_i(x) = \lambda y_i(x), \quad \text{on } [0, l_i], \quad (2.1)$$

where $y_i$ and $q_i$ are the restrictions of $y$ and $q$, respectively, to the edge $e_i$.

Let $t = \frac{x}{l_i}$ and $\tilde{y}_i(t) = \frac{1}{\sqrt{l_i}}y_i(l_it)$. Then, for each $i = 1, \ldots, K$, (2.1) transforms to the equation

$$-\tilde{y}''_i(t) + l_i^2(Q_i - \lambda)\tilde{y}_i = 0, \quad t \in [0, 1], \quad (2.2)$$

where $Q_i(t) = q_i(l_it)$. The equations (2.2) for $i = 1, \ldots, K$, are equivalent to the system

$$LY := -W\tilde{Y}'' + Q\tilde{Y} = \lambda\tilde{Y}, \quad (2.3)$$

on $[0, 1]$, where $W = \text{diag}[l_1^{-2}, \ldots, l_K^{-2}]$, $Q = \text{diag}[Q_1, \ldots, Q_K]$ and $\tilde{Y} = [\tilde{y}_1, \ldots, \tilde{y}_K]^T$.

The boundary conditions on the graph may be written as

$$\sum_{j=1}^K [\alpha_{ij}y_j(0) + \beta_{ij}y_j'(0)] + \sum_{j=1}^K [\gamma_{ij}y_j(l_j) + \delta_{ij}y_j'(l_j)] = 0, \quad i = 1, \ldots, 2K, \quad (2.4)$$

where $N$ is the total number of linearly independent boundary conditions, see [9]. Under the above mapping these boundary conditions transform to

$$\tilde{A}\tilde{Y}(0) + \tilde{B}\tilde{Y}'(0) + \tilde{C}\tilde{Y}(1) + \tilde{D}\tilde{Y}'(1) = 0, \quad (2.5)$$

where $\tilde{A} = [\sqrt{l_j}\alpha_{ij}]$, $\tilde{B} = [\sqrt{l_j}\beta_{ij}]$, $\tilde{C} = [\sqrt{l_j}\gamma_{ij}]$ and $\tilde{D} = [\sqrt{l_j}\delta_{ij}]$.

Let $L^2_K$ denote the weighted vector $L^2$-space

$$L^2_K = \{ F : [0, 1] \to \mathbb{C}^K \mid F_i \in L^2[0, 1], i = 1, \ldots, K \}$$
with the inner product
\[< F, G >_W = \sum_{i=1}^{K} l_i^2 \int_0^1 F_i \overline{G_i} \, dt = \int_0^1 F^T W^{-1} G \, dt. \]

Here we note that \(< \tilde{Y}, \tilde{Z} >_W = (y, z)_G \) and \(< L \tilde{Y}, \tilde{Z} >_W = (ly, z)_G \). Thus the boundary value problem on the graph is formally self-adjoint in \( L^2(G) \) if and only if the system boundary value problem, (2.3), (2.5), is formally self-adjoint in \( L^2_K \). The mapping, \( \psi : y \mapsto \tilde{Y} \), with \( \psi : L^2(G) \to L^2_K \) is an isometry, and the representation of \( l \) in \( L^2_K \) is \( \psi l \psi^{-1} = L \) where \( L \) is as given by (2.3), (2.5).

In [9], it was shown that the formally self-adjoint boundary value problem, (2.3), (2.5), is equivalent to a formally self-adjoint boundary value problem of dimension \( 2K \) with separated boundary conditions, i.e., is equivalent to a system of the form

\[-MY'' + PY = \lambda Y, \tag{2.6} \]

with boundary conditions

\[ A^* Y(0) - B^* Y'(0) = 0, \tag{2.7} \]
\[ \Gamma^* Y(1) - \Delta^* Y'(1) = 0, \tag{2.8} \]

where \( M = 4 \text{diag} \left[ \frac{1}{l_1}, \ldots, \frac{1}{l_K}, \frac{1}{l_1}, \ldots, \frac{1}{l_K} \right] \), \( P \) is a diagonal matrix dependent on the potential on each edge of the graph, \( A^* = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -I \\ 0 & 0 \end{bmatrix} \), \( B^* = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 \\ -I & -I \end{bmatrix} \). Here, \( \Gamma^* = [\bar{C} \bar{A}] \) and \( \Delta^* = 2[-\bar{D} \bar{B}] \), are \( 2K \times 2K \) constant matrices.

The boundary value problem (2.6)-(2.8) can be rewritten as the first order system

\[ Y' = Z \quad \text{and} \quad Z' = -G(x)Y, \tag{2.9} \]

with boundary conditions \( A^* Y(0) - B^* Z(0) = 0 = \Gamma^* Y(1) - \Delta^* Z(1) \), where \( G(x) = M^{-1}(\lambda - P(x)) \). Without loss of generality it may be assumed that the following three properties hold:

(A) \( G(x) \) is continuous, real valued and symmetric.
(B) \( A^* B = B^* A \) and \( \Gamma^* \Delta = \Delta^* \Gamma \).
(C) \( A^* A + B^* B = I \) and \( \Gamma^* \Gamma + \Delta^* \Delta = I \).

Here property (A) follows directly from the nature of \( M \) and \( P \). For formally self-adjoint boundary conditions it was shown in [9, Lemma 7.1] that (B) and (C) do not pose additional constraints.

In order to define the M-matrix, in [11], we needed the two solutions, \( W_2 \) and \( W_3 \), of (2.6) such that

\[ W(x) = \begin{bmatrix} W_2(x) & W_3(x) \\ W_2'(x) & W_3'(x) \end{bmatrix}, \tag{2.10} \]
obey the terminal condition
\[ W(1) = R, \quad (2.11) \]
where \( R = \begin{bmatrix} -\Gamma & \Delta \\ \Delta & \Gamma \end{bmatrix} \).

The Titchmarsh-Weyl M-matrix, \( \mathcal{M} = \mathcal{M}(\lambda) \), of (2.6)-(2.8) was defined, in [11], to be the matrix \( \mathcal{M} \) given by
\[ \Psi = W_2 + W_3 \mathcal{M}, \quad (2.12) \]
with the constraint that \( \Psi \) obeys (2.7).

## 3 The nature of the poles of the M-matrix

Here we use Wronskian identities and the Green’s function to study the nature of the poles of the M-matrix. In particular in Theorem 3.3 we show that the poles of the M-matrix are simple, located on the real axis and are the eigenvalues of (2.6)-(2.8). In addition we show that the residue at a pole is a negative semi-definite matrix of rank equal to the multiplicity of the eigenvalue.

**Lemma 3.1** For \( \lambda \in \mathbb{R} \), the following Wronskian identities hold:
\[
\begin{align*}
W_3''(x)W_2(x) - W_3'(x)W_2'(x) & = -I, \\
W_2''(x)W_3'(x) - W_2'(x)W_3''(x) & = -I, \\
W_2''(x)W_3(x) - W_2'(x)W_3'(x) & = I, \\
W_3''(x)W_2'(x) - W_3'(x)W_2''(x) & = I, \\
W_2''(x)W_2(x) - W_2'(x)W_2'(x) & = 0, \\
W_3''(x)W_3(x) - W_3'(x)W_3'(x) & = 0.
\end{align*}
\]

*Proof:* We prove only the first two identities, the proofs of the other four, being similar, are omitted. We begin by showing that the matrix-Wronskians \( W_3''(x)W_2(x) - W_3'(x)W_2'(x) \) and \( W_2''(x)W_3(x) - W_2'(x)W_3'(x) \) are constant. Observe that
\[
[W_3''(x)W_2(x) - W_3'(x)W_2'(x)]' = W_3'''(x)W_2(x) - W_3''(x)W_2''(x), \quad (3.1)
\]
and, since \( W_3(x) \) and \( W_2(x) \) are solutions of (2.6),
\[
-W_3'''(x)M + W_3''(x)P = \bar{\lambda}W_3'''(x) \quad \text{and} \quad -W_2'''(x)M + W_2''(x)P = \bar{\lambda}W_2'''(x). \quad (3.2)
\]
Substituting (3.2) into (3.1) we obtain
\[
[W_3''(x)W_2(x) - W_3'(x)W_2'(x)]' = W_3''(x)(P - \bar{\lambda})M^{-1}W_2(x) - W_3'(x)M^{-1}(P - \lambda)W_2(x).
\]
Now, since \( M^{-1} \) and \( P - \lambda \) are diagonal matrices and \( \lambda \in \mathbb{R} \), \( (P - \lambda)M^{-1} = M^{-1}(P - \lambda) \). Therefore \( [W_3^o(x)W_2(x) - W_3^o(x)W'_2(x)]' = 0 \), and \( W_3^o(x)W_2(x) - W_3^o(x)W'_2(x) \) is constant.

Now \( W_3^o(1)W_2(1) - W_3^o(1)W'_2(1) = -\Gamma^*\Gamma - \Delta^*\Delta = -I \). Thus
\[
W_3^o(x)W_2(x) - W_3^o(x)W'_2(x) = -I.
\]

Taking the Hermitian conjugate of the above equation gives
\[
W_2^o(x)W_3^o(x) - W_2^o(x)W_3^o(x) = -I. \quad \blacksquare
\]

**Proposition 3.2** The Green’s function for the boundary value problem (2.6)-(2.8) can be represented as
\[
G(x,t) = \begin{cases} 
W_3(x)\Psi(t)M^{-1}, & t < x \\
\Psi(x)W_3^o(t)M^{-1}, & t > x 
\end{cases} \tag{3.3}
\]

**Proof:** From [8, p. 24-25] the Green’s function of (2.6)-(2.8) exists. It thus remains for us to obtain (3.3). Since \( G(x,t) \) is a solution of (2.6) with respect to \( x \) for each \( x \neq t \), we may assume
\[
G(x,t) = \begin{cases} 
W_2(x)U_1(t) + W_3(x)U_2(t), & t < x \\
W_2(x)H_1(t) + W_3(x)H_2(t), & t > x, 
\end{cases}
\]
where \( U_1, U_2, H_1 \) and \( H_2 \) must be determined. Let
\[
Z(x) := \int_0^1 G(x,t)F(t) \, dt 
\]
\[
= W_2(x) \int_0^x U_1(t)F \, dt + W_3(x) \int_0^x U_2(t)F \, dt + W_2(x) \int_x^1 H_1(t)F \, dt + W_3(x) \int_x^1 H_2(t)F \, dt.
\]
By definition, \( Z(x) \) is a solution of
\[
-MZ'' + (P - \lambda)Z = F, \quad F \in \mathcal{L}^2[0,1]. \tag{3.4}
\]
Differentiating \( Z(x) \in H^2[0,1] \) we obtain
\[
Z'(x) = W_2'(x) \int_0^x U_1(t)F \, dt + W_2'(x) \int_0^x U_2(t)F \, dt + W_2'(x) \int_x^1 H_1(t)F \, dt 
\]
\[
+ W_3'(x) \int_x^1 H_2(t)F \, dt + W_2(x)U_1(x)F + W_3(x)U_2(x)F - W_2(x)H_1(x)F
\]
\[
- W_3(x)H_2(x)F
\]
which, since \( Z'(x) \in H^1[0,1] \) for all \( F \in \mathcal{L}^2[0,1] \), gives the condition
\[
W_2(x)U_1(x) + W_3(x)U_2(x) - W_2(x)H_1(x) - W_3(x)H_2(x) = 0. \tag{3.5}
\]
By substituting \( Z''(x) \) into (3.4), since \( F \) may vary over all of \( \mathcal{L}^2[0,1] \), we get the condition

\[
W'_2(x)U_1(x) + W'_3(x)U_2(x) - W'_2(x)H_1(x) - W'_3(x)H_2(x) = -M^{-1}. \tag{3.6}
\]

Since \( Z \) must obey the boundary condition at \( x = 1 \), substituting \( Z(1) \) and \( Z'(1) \) into (2.8) and using (2.11) together with properties (B) and (C) of section 2 we get that

\[
\int_0^1 U_1(t)F dt = 0,
\]

for all \( F \in \mathcal{L}^2[0,1] \). Thus \( U_1 \equiv 0 \).

Similarly substituting \( Z(0) \) and \( Z'(0) \) into (2.7) we obtain

\[
(A^*W_2(0) - B^*W'_2(0))H_1(t) = -(A^*W_3(0) - B^*W'_3(0))H_2(t)
\]
a.e. on \([0,1]\). Which, by [11, equation (2.15)], can be rewritten as

\[
(B^*W'_3(0) - A^*W_3(0))\mathcal{M}(\lambda)H_1(t) = (B^*W'_3(0) - A^*W_3(0))H_2(t),
\]
a.e. on \([0,1]\). Since \( B^*W'_3(0) - A^*W_3(0) \) is invertible away from the eigenvalues

\[
\mathcal{M}H_1(t) = H_2(t),
\]
a.e. on \([0,1]\) for \( \lambda \) not an eigenvalue.

Thus (3.5) and (3.6) become

\[
W_3(x)U_2(x) - W_2(x)H_1(x) - W_3(x)\mathcal{M}H_1(x) = 0 \tag{3.7}
\]
and

\[
W'_3(x)U_2(x) - W'_2(x)H_1(x) - W'_3(x)\mathcal{M}H_1(x) = -M^{-1}, \tag{3.8}
\]
respectively.

Multiplying (3.7) by \( W'_2(x) \) and (3.8) by \( W'_2(x) \) and subtracting the resulting equations gives

\[
U_2 - (W''_2W_2 - W'_2W'_2)H_1 - (W''_3W_3 - W'_3W'_3)\mathcal{M}H_1 = W'_2M^{-1}
\]
a.e. on \([0,1]\) which by Lemma 3.1 gives

\[
U_2(x) = \mathcal{M}H_1(x) + W'_2(x)M^{-1}.
\]

Similarly multiplying (3.7) by \( W'_3(x) \) and (3.8) by \( W'_3(x) \), subtracting the resulting equations and using Lemma (3.1) gives

\[
H_1(x) = W'_3(x)M^{-1}.
\]
Thus
\[ G(x,t) = \begin{cases} W_3(x)(\mathcal{M}W_3^*(t) + W_2^*(t))M^{-1}, & t < x \\ (W_2(x) + W_3(x)\mathcal{M})W_3^*(t)M^{-1}, & t > x \end{cases}. \]

For \( \lambda \in \mathbb{R} \), \( \mathcal{M}(\lambda) = \mathcal{M}^*(\lambda) \) since \( \mathcal{M}(\lambda) \) is a Herglotz function, see [11]. So by (2.12) we get
\[ G(x,t) = \begin{cases} W_3(x)\Psi^*(t)M^{-1}, & t < x \\ \Psi(x)W_3^*(t)M^{-1}, & t > x \end{cases}. \]

In [11] it was shown that the M-matrix defined in (2.12) is a Herglotz function and as such it admits the following representation, see [17],
\[ \mathcal{M}(\lambda) = \mathcal{C} + \mathcal{D}\lambda + \sum_{\lambda_n} M_n \left( \frac{1}{\lambda_n - \lambda} - \frac{\lambda_n}{1 + \lambda_n^2} \right), \]
where \( \mathcal{C} = \text{Re}(\mathcal{M}(i)) \) and \( \mathcal{D} = \lim_{\eta \to \infty} (\frac{1}{i\eta} \mathcal{M}(i\eta)) \geq 0 \). Thus
\[
\lim_{\lambda \to \lambda_n} (\lambda - \lambda_n)\mathcal{M}(\lambda) = \lim_{\lambda \to \lambda_n} (\lambda - \lambda_n)(\mathcal{C} + \mathcal{D}\lambda) + \lim_{\lambda \to \lambda_n} (\lambda - \lambda_n) \sum_{\lambda_n} M_n \left( \frac{1}{\lambda_n - \lambda} - \frac{\lambda_n}{1 + \lambda_n^2} \right)
\]
\[ = -M_n. \]

We now give the main theorem of this section.

**Theorem 3.3** The poles of the M-matrix are simple, located on the real axis and are the eigenvalues of (2.6)-(2.8). The residue at a pole is a negative semi-definite matrix of rank equal to the multiplicity of the eigenvalue.

**Proof:** By [17], since \( \mathcal{M}(\lambda) \) is a matrix-valued Herglotz function, it follows that the poles of the M-matrix are simple and located on the real axis and that the residue at a pole is a negative semi-definite matrix. Also, from [11, Proposition 2.2], all the poles of \( \mathcal{M} \) are eigenvalues of (2.6)-(2.8). At an eigenvalue of (2.6)-(2.8) the Green’s function of (2.6)-(2.8) has a pole, see representation (3.12), giving that if \( \lambda \) is an eigenvalue of (2.6)-(2.8) then \( \lambda \) is a pole of \( G(x,t) \) and thus, by (3.3), \( \lambda \) is a pole of \( \Psi \) and hence, by (2.12), is a pole of \( \mathcal{M} \). Therefore it remains to show that \(-M_n\), the residue of the pole of \( \mathcal{M}(\lambda) \) at \( \lambda_n \), has rank equal to the multiplicity of the eigenvalue.

From (2.12)
\[ (\lambda - \lambda_n)\Psi(x,\lambda) = (\lambda - \lambda_n)W_2(x,\lambda) + (\lambda - \lambda_n)W_3(x,\lambda)\mathcal{M}(\lambda) \]
and taking the limit as \( \lambda \) tends to \( \lambda_n \) gives
\[
\lim_{\lambda \to \lambda_n} (\lambda - \lambda_n)\Psi(x,\lambda) = W_3(x,\lambda_n) \lim_{\lambda \to \lambda_n} (\lambda - \lambda_n)\mathcal{M}(\lambda). \]
Thus
\[ \lim_{\lambda \to \lambda_n} (\lambda - \lambda_n)\Psi(x, \lambda) = -W_3(x, \lambda_n)M_n. \] (3.9)

By taking the Hermitian transpose of (3.9) we obtain
\[ \lim_{\lambda \to \lambda_n} (\lambda - \lambda_n)\Psi^*(x, \lambda) = -M_n^*W_3^*(x, \lambda_n). \] (3.10)

By proposition 3.2 we have that the Green’s function for the boundary value problem (2.6)-(2.8) can be represented as
\[ G(x, t) = \begin{cases} W_3(x, \lambda)\Psi^*(t, \lambda)M^{-1}, & t < x \\ \Psi(x, \lambda)W_3^*(t, \lambda)M^{-1}, & t > x. \end{cases} \]

Thus using (3.9) and (3.10)
\[
\lim_{\lambda \to \lambda_n} (\lambda - \lambda_n)G(x, t) = \lim_{\lambda \to \lambda_n} \begin{cases} (\lambda - \lambda_n)W_3(x, \lambda)\Psi^*(t, \lambda)M^{-1}, & t < x \\ (\lambda - \lambda_n)\Psi(x, \lambda)W_3^*(t, \lambda)M^{-1}, & t > x \end{cases} = \begin{cases} W_3(\lambda_n, x)(\lim_{\lambda \to \lambda_n} (\lambda - \lambda_n)\Psi^*(t, \lambda))M^{-1}, & t < x \\ (\lim_{\lambda \to \lambda_n} (\lambda - \lambda_n)\Psi(x, \lambda))W_3^*(t, \lambda_n)M^{-1}, & t > x \end{cases} = -\begin{cases} W_3(\lambda_n, x)M_n^*W_3^*(t, \lambda_n)M^{-1}, & t < x \\ W_3(\lambda_n, x)M_nW_3^*(t, \lambda_n)M^{-1}, & t > x. \end{cases}
\]

Hence
\[ \lim_{\lambda \to \lambda_n} (\lambda - \lambda_n)G(x, t) = -W_3(\lambda_n, x)M_nW_3^*(t, \lambda_n)M^{-1} \] (3.11)
for all \( t \) since, as \( \mathcal{M}(\lambda) \) is a Herglotz function, \( \mathcal{M}^*(\lambda) = \mathcal{M}(\lambda) \), for \( \lambda \in \mathbb{R} \), giving \( M_n^* = M_n \).

We now observe that, by [8], the Green’s function of the boundary value problem (2.6)-(2.8) is also given by
\[
\int_0^1 G(x, t)F(t)\,dt = \sum_{n=1}^{\infty} \sum_{j=1}^{\nu_n} \frac{<F, F_{n,j}>}{\lambda_n - \lambda} F_{n,j}(x), \quad (3.12)
\]
for \( F \in \mathcal{L}^2[0, 1] \), where \( \nu_n \) is the multiplicity of the eigenvalue \( \lambda_n \), \( F_{n,j} \), \( j = 1, \ldots, \nu_n \), is an orthonormal sequence of eigenfunctions corresponding to \( \lambda_n \) (\( \lambda_n \) not repeated according to multiplicity). Recall that the inner product is given by
\[
<F, G> = \sum_{i=1}^{2K} \int_0^1 F_iG_i \,dt = \int_0^1 F^TM^{-1}G\,dt,
\] (3.13)
where \( l_i = l_{K+i} \) for \( i = 1, \ldots, K \).

Since \( F, F_{n,j} \) are column vectors and \( M \) is a diagonal matrix \( F^TM^{-1}F_{n,j}F_{n,j}(x) = F_{n,j}(t)M^{-1}F(t)F_{n,j}(x) \). But \( F_{n,j}^*M^{-1}F(t) \) is scalar so \( F_{n,j}^*M^{-1}F(t) \) and \( F_{n,j}(x) \) commute to give
\[
\int_0^1 G(x, t)F(t)\,dt = \int_0^1 \sum_{n=1}^{\infty} \sum_{j=1}^{\nu_n} \frac{1}{\lambda_n - \lambda} F_{n,j}(x)F_{n,j}^*M^{-1}F(t)\,dt,
\]

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for all $F \in \mathcal{L}^2[0,1]$. Hence,

$$G(x,t) = \sum_{n=1}^{\infty} \sum_{j=1}^{\nu_n} \frac{1}{\lambda_n - \lambda} F_{n,j}(x) F_{n,j}^*(t) M^{-1}.$$ 

Taking the limit as $\lambda \to \lambda_n$ we get

$$\lim_{\lambda \to \lambda_n} (\lambda - \lambda_n) G(x,t) = - \sum_{j=1}^{\nu_n} F_{n,j}(x) F_{n,j}^*(t) M^{-1}.$$ 

Thus by (3.11)

$$\sum_{j=1}^{\nu_n} F_{n,j}(x) F_{n,j}^*(t) M^{-1} = W_3(\lambda_n, x) M_n W_3^*(t, \lambda_n) M^{-1}. \quad (3.14)$$

Since $F_{n,j}$ is an eigenfunction corresponding to $\lambda_n$, it can be written as

$$F_{n,j}(x) = W_3(\lambda_n, x) c_{n,j} \quad (3.15)$$

where $c_{n,j}$ is a column vector. Substituting (3.15) into (3.14) gives

$$W_3(\lambda_n, x) \left[ \sum_{j=1}^{\nu_n} c_{n,j} c_{n,j}^* - M_n \right] W_3^*(\lambda_n, t) = 0. \quad (3.16)$$

Differentiating (3.16) with respect to $x$ we obtain

$$W_3'(\lambda_n, x) \left[ \sum_{j=1}^{\nu_n} c_{n,j} c_{n,j}^* - M_n \right] W_3^*(\lambda_n, t) = 0. \quad (3.17)$$

Pre-multiplying (3.16) by $W_2^*(\lambda_n, x)$ and (3.17) by $W_2^*(\lambda_n, x)$ and subtracting the resulting equations gives

$$W_2^*(\lambda_n, x) W_3(\lambda_n, x) \left[ \sum_{j=1}^{\nu_n} c_{n,j} c_{n,j}^* - M_n \right] W_3^*(\lambda_n, t)$$

$$- W_2^*(\lambda_n, x) W_3'(\lambda_n, x) \left[ \sum_{j=1}^{\nu_n} c_{n,j} c_{n,j}^* - M_n \right] W_3^*(\lambda_n, t) = 0.$$ 

Now applying Lemma 3.1 we obtain

$$\left[ \sum_{j=1}^{\nu_n} c_{n,j} c_{n,j}^* - M_n \right] W_3^*(\lambda_n, t) = 0. \quad (3.18)$$
Differentiating (3.18) with respect to $t$ gives

$$\left[ \sum_{j=1}^{\nu_n} c_{n,j} c_{n,j}^* - M_n \right] W_3^*(\lambda_n, t) = 0.$$  \hspace{1cm} (3.19)

Pre-multiplying (3.18) by $W_2^*(\lambda_n, x)$ and (3.19) by $W_2^*(\lambda_n, t)$, subtracting the resulting equations and using lemma 3.1 gives

$$\sum_{j=1}^{\nu_n} c_{n,j} c_{n,j}^* = M_n,$$

and as already noted $-M_n$ is a negative semi-definite matrix.

Now define the $2K \times 2K$ matrix

$$C_n := [c_{n,1}, \ldots, c_{n,\nu_n}, 0, \ldots, 0].$$

Then since $c_{n,1}, \ldots, c_{n,\nu_n}$ are linearly independent the rank of $C_n$ is $\nu_n$. Also the rank of $C_n$ is equal to the number of non-zero eigenvalues of $C_n$, counted by multiplicity. Denote these eigenvalues by $\mu_1, \ldots, \mu_{\nu_n}$. Then $C_n C_n^*$ has non-zero eigenvalues $|\mu_1|^2, \ldots, |\mu_{\nu_n}|^2$. Thus $C_n C_n^*$ has rank $\nu_n$ and $\text{Rank}(-M_n) = \nu_n$. \hfill $\blacksquare$

### 4 Norming constants and spectral measure

In this section we obtain expressions for the norming constants associated with the boundary value problem (2.6)-(2.8). We then give a form for the spectral measure in terms of the norming constants and show how it relates to the $M$-matrix.

We define the norming constant $H_n$ by

$$H_n = \int_0^1 W_3^*(\lambda_n, x) M^{-1} W_3(\lambda_n, x) dx.$$

It should be noted that the norming constants given here are consistent with those found using methods analogous to those of Freiling and Yurko, in [14].

From (3.15) and the definition of $F_{n,j}$ we have that

$$< W_3(\lambda_n, x) c_{n,j} , W_3(\lambda_n, x) c_{n,i} > = \delta_{i,j},$$

which, by (3.13), implies that

$$\int_0^1 c_{n,j}^T W_3^T(\lambda_n, x) M^{-1} W_3(\lambda_n, x) c_{n,i} dx = \delta_{i,j}.$$
Hence
\[ c_{n,j}^* H_n c_{n,i} = c_{n,j}^* \left[ \int_0^1 W_3^*(\lambda_n, x) M^{-1} W_3(\lambda_n, x) \, dx \right] c_{n,i} = \delta_{i,j}. \] (4.1)

So if \( C_n = [c_n, 1, \ldots, c_n, \nu_n] \) then
\[ C_n^* H_n C_n = I_{\nu_n}. \]

Also note that \( \langle W_3(\lambda_n, x)c_{n,j}, W_3(\lambda_m, x)c_{m,k} \rangle = 0 \) for all \( n \neq m \).

**Theorem 4.1** Let the resolution function, \( \rho \), be given by \( \lim_{\mu \to -\infty} \rho(\mu) = 0 \) and
\[ \rho(\mu) - \rho(\mu_1) = \frac{1}{2\pi i} \lim_{\delta \to 0^+} \int_{\mu_1}^{\mu_2} (\mathcal{M}(u + i\delta) - \mathcal{M}^*(u + i\delta)) \, du, \] (4.2)
where \( \mu_1 < \mu_2 \) are not eigenvalues of (2.6)-(2.8). Then
\[ \rho(\mu) = -\sum_{\lambda_i < \mu} \text{Res}_{\lambda=\lambda_i}(\mathcal{M}(\lambda)) \]
and \( \rho \) is the spectral measure associated with (2.6)-(2.8), i.e.
\[ \rho(\mu) = \sum_{\lambda_i < \mu} M_i \]
with
\[ f(x) = \int_{-\infty}^{\infty} \left( W_3(\lambda, x) d\rho(\lambda) \int_0^1 W_3^*(\lambda, \tau) M^{-1} f(\tau) \, d\tau, \right), \] (4.3)
for all \( f \in L^2[0,1] \).

**Proof:** For \( \mu \in \mathbb{R} \), since \( \mathcal{M}(\mu) \) is a Herglotz function, \( \mathcal{M}^*(\mu) = \mathcal{M}(\mu) \). Thus if there is no eigenvalue in \( [\mu_1, \mu_2] \) we obtain
\[ \lim_{\delta \to 0^+} \int_{\mu_1}^{\mu_2} (\mathcal{M}(u + i\delta) - \mathcal{M}^*(u + i\delta)) \, du = \int_{\mu_1}^{\mu_2} (\mathcal{M}(u) - \mathcal{M}^*(u)) \, du = 0, \]
as expected for a spectral measure. Since the spectrum is bounded below, this also gives that \( \rho(\mu) = 0 \) for all \( \mu < \lambda_0 \).

Note that since \( \mathcal{M}(z) \) is a Herglotz function \( \mathcal{M}(z) = \mathcal{M}^*(\bar{z}) \), for \( z \in \mathbb{C} \), giving that \( \mathcal{M}(u + i\delta) = \mathcal{M}^*(u - i\delta) \). Hence for a single eigenvalue of (2.6)-(2.8), \( \lambda_0 \in (\mu_1, \mu_2) \), we have
\[ \lim_{\delta \to 0^+} \int_{\mu_1}^{\mu_2} (\mathcal{M}(u + i\delta) - \mathcal{M}^*(u + i\delta)) \, du = \lim_{\delta \to 0^+} \int_{\mu_1}^{\mu_2} (\mathcal{M}(u + i\delta) - \mathcal{M}(u - i\delta)) \, du \]
\[ = \int_{C_1} \mathcal{M}(\lambda_0 + z) \, dz \]
\[ = -\int_{C_2} \mathcal{M}(\lambda_0 + z) \, dz \]
\[ = -2\pi i \text{Res}_{\lambda=\lambda_0}(\mathcal{M}(\lambda)) \]
where \( C_1 \) and \( C_2 \) are as given in the diagram below.
Therefore, for general $\mu$ not an eigenvalue we get

$$\rho(\mu) = \sum_{\lambda_i < \mu} -\text{Res}_{\lambda = \lambda_i} (M(\lambda)) = \sum_{\lambda_i < \mu} M_i.$$ 

Now by (4.1),

$$c_{n,j}^* \left[ \int_0^1 W_3^*(\lambda_n, x)M^{-1}W_3(\lambda_m, x) \, dx \right] c_{m,i} = \int_0^1 (W_3(\lambda_n, x)c_{n,j})^*M^{-1}(W_3(\lambda_m, x)c_{m,i}) \, dx$$

$$= \delta_{i,j}\delta_{m,n}. \quad (4.4)$$

Since $(W_3(\lambda_n, \tau)c_{n,j})^*M^{-1}f(\tau)$ is a scalar, the spectral projection $E_{\lambda_n}[f]$ of $f$ is given by

$$E_{\lambda_n}[f](x) = \sum_{j=1}^{\nu_n} \int_0^1 (W_3(\lambda_n, \tau)c_{n,j})^*M^{-1}f(\tau) \, d\tau W_3(\lambda_n, x)c_{n,j}$$

$$= \sum_{j=1}^{\nu_n} W_3(\lambda_n, x)c_{n,j}c_{n,j}^* \int_0^1 W_3^*(\lambda_n, \tau)M^{-1}f(\tau) \, d\tau$$

for all $f \in L^2[0,1]$. Thus

$$E_{\lambda_n}[f](x) = W_3(\lambda_n, x) \sum_{j=1}^{\nu_n} c_{n,j}c_{n,j}^* \int_0^1 W_3^*(\lambda_n, \tau)M^{-1}f(\tau) \, d\tau$$

giving

$$E_{\lambda_n}[f](x) = W_3(\lambda_n, x)M_n \int_0^1 W_3^*(\lambda_n, \tau)M^{-1}f(\tau) \, d\tau. \quad (4.5)$$

Hence

$$f(x) = \sum_{n=0}^{\infty} W_3(\lambda_n, x)M_n \int_0^1 W_3^*(\lambda_n, \tau)M^{-1}f(\tau) \, d\tau$$

and thus (4.3) holds for all $f \in L^2[0,1]$.  

\section{Recovery of the operator}

In [26], Marčenko considered an inverse spectral problem for scalar Sturm-Liouville boundary value problems. In this, the main section of the paper, we build on the
method of Marčenko to recover the potential for Sturm-Liouville boundary value problems on a graph from the M-matrix. Boundary conditions are recovered using M-matrix asymptotics, see [11].

From Marčenko, [26, p. 30, Theorem 1.2.2] we have as a direct consequence the following lemma:

**Lemma 5.1** There exists a kernel, \( k_{h,m,q}(t,y) \), \( (k_{\infty,m,q}(t,y) \) resp.) such that,

\[
v_{h,m,q}[f](t) := \int_1^t k_{h,m,q}(t,y)f(y) \, dy, \quad (v_{\infty,m,q}[f](t) := \int_1^t k_{\infty,m,q}(t,y)f(y) \, dy \) resp.
\]
defines a continuous linear transformation on \( \mathcal{L}^2[0,1] \), and if \( y_\lambda \) is the solution of \(-my_\lambda'' = \lambda y_\lambda \) on \([0,1] \) with \( y'_\lambda(1) = hy_\lambda(1) \) \( (y_\lambda(1) = 0 \) resp.), for \( m > 0 \) a real constant, then

\[
z_\lambda := (I + v_{h,m,q})y_\lambda \quad (z_\lambda := (I + v_{\infty,m,q})y_\lambda \) resp.
\]
is the solution of \(-mqz''_\lambda + qz_\lambda = \lambda z_\lambda \) on \([0,1] \) with \( z'_\lambda(1) = hy_\lambda(1) \) and \( z_\lambda(1) = y_\lambda(1) \)

\[
(z_\lambda(1) = 0 \) and \( z'_\lambda(1) = y'_\lambda(1) \) resp.), for each \( \lambda \in \mathbb{R} \).

Let \((\Gamma^*, \Delta^*, P)\) denote the boundary value problem (2.6)-(2.8) and \((\tilde{\Gamma}^*, \tilde{\Delta}^*, \tilde{P})\) the boundary value problem (2.6)-(2.8) but with \( \Gamma \) replaced by \( \tilde{\Gamma} \), \( \Delta \) by \( \tilde{\Delta} \) and \( P \) by \( \tilde{P} \).

**Lemma 5.2** Let the problems \((\Gamma^*, \Delta^*, P)\) and \((\tilde{\Gamma}^*, \tilde{\Delta}^*, \tilde{P})\) have the same M-matrix. If there exists a linear continuous transformation operator, \( H \), on \( \mathcal{L}^2[0,1] \), independent of \( \lambda \), which maps

\[
H[W_3(\lambda, x)] = \tilde{W}_3(\lambda, x),
\]
where \( \tilde{W}_3(\lambda, x) \) is the solution to \((\tilde{\Gamma}^*, \tilde{\Delta}^*, \tilde{P})\) obeying \( \tilde{W}_3(\lambda, 1) = \tilde{\Delta} \) and \( \tilde{W}_3'(\lambda, 1) = \tilde{\Gamma} \), then \( H \) is unitary.

**Proof:** Consider \( f \in \mathcal{L}^2[0,1], \) then as in (4.3),

\[
H[f](x) = \sum_{n=1}^{\infty} H \left[ W_3(\lambda_n, x) M_n \int_0^1 W_3^*(\lambda_n, \tau) M^{-1} f(\tau) \, d\tau \right]
\]

\[
= \sum_{n=1}^{\infty} H[W_3(\lambda_n, x)] M_n \int_0^1 W_3^*(\lambda_n, \tau) M^{-1} f(\tau) \, d\tau
\]

\[
= \sum_{n=1}^{\infty} H[W_3(\lambda_n, x)] \sum_{j=1}^{\nu_n} c_{n,j} c_{n,j}^* \int_0^1 W_3^*(\lambda_n, \tau) M^{-1} f(\tau) \, d\tau
\]

\[
= \sum_{n=1}^{\infty} \tilde{W}_3(\lambda_n, x) \sum_{j=1}^{\nu_n} c_{n,j} c_{n,j}^* \int_0^1 W_3^*(\lambda_n, \tau) M^{-1} f(\tau) \, d\tau.
\]
and $H[f] \in L^2[0,1]$. Since $\mathcal{M}(\lambda) = \tilde{\mathcal{M}}(\lambda)$ for all $\lambda \in \mathbb{R}$, we have
\[
\sum_{j=1}^{\nu_n} c_{n,j}^* c_{n,j} = M_n = \tilde{M}_n = \sum_{j=1}^{\nu_n} \tilde{c}_{n,j} \tilde{c}_{n,j}^*.
\]

Hence
\[
\int_0^1 (H[f])^* M^{-1} H[f] \, dx = \int_0^1 \sum_{m,n=1}^{\infty} \left( \int_0^1 W_3(\lambda_n, \tau) M^{-1} f(\tau) \, d\tau \right)^* c_{n,j}^* \tilde{W}_3^*(\lambda_n, x) 
\times M^{-1} \tilde{W}_3(\lambda_m, x) \sum_{i=1}^{\nu_n} c_{m,i}^* c_{m,i} \int_0^1 W_3(\lambda_m, s) M^{-1} f(s) \, ds \, dx
\]
\[
= \sum_{m,n=1}^{\infty} \left( \int_0^1 W_3(\lambda_n, \tau) M^{-1} f(\tau) \, d\tau \right)^* \sum_{j=1}^{\nu_n} \tilde{c}_{n,j} \tilde{c}_{n,j}^* \int_0^1 W_3^*(\lambda_n, x) M^{-1} \tilde{W}_3(\lambda_n, x) \, dx
\times \sum_{i=1}^{\nu_m} \tilde{c}_{m,i} c_{m,i} \int_0^1 W_3(\lambda_m, s) M^{-1} f(s) \, ds.
\]

Now (4.4) also holds for $c_{n,j}$ replaced by $\tilde{c}_{n,j}$ and $W_3(\lambda_n, x)$ by $\tilde{W}_3(\lambda_n, x)$, so
\[
\tilde{c}_{n,j} \int_0^1 \tilde{W}_3^*(\lambda_n, x) M^{-1} \tilde{W}_3(\lambda_n, x) \, dx \tilde{c}_{m,i} = \delta_{i,j} \delta_{m,n}
\]
which, along with $M_n = \tilde{M}_n$, gives
\[
\int_0^1 (H[f])^* M^{-1} H[f] \, dx = \sum_{n=1}^{\infty} \sum_{j=1}^{\nu_n} \left( \int_0^1 W_3(\lambda_n, \tau) M^{-1} f(\tau) \, d\tau \right)^* c_{n,j} c_{n,j}^* \int_0^1 W_3(\lambda_n, s) M^{-1} f(s) \, ds
\]
\[
= ||f||^2.
\]

Therefore $||H[f]||^2 = ||f||^2$ for arbitrary $f \in L^2[0,1]$. Hence $H$ is unitary in $L^2[0,1]$. \hfill \blacksquare

We will now use Lemma 5.1 and Lemma 5.2 to prove the main result of the paper.

**Theorem 5.3** If the problems $(\Gamma^*, \Delta^*, P)$ and $(\tilde{\Gamma}^*, \tilde{\Delta}^*, \tilde{P})$ have the same $M$-matrix, i.e. $\mathcal{M}(\lambda) = \tilde{\mathcal{M}}(\lambda)$ for all $\lambda \in \mathbb{R}$, then $\Delta = U \tilde{\Delta}$ and $\Gamma = U \tilde{\Gamma}$. Here $U = \Gamma \tilde{\Gamma}^* + \Delta \tilde{\Delta}^*$ is a unitary matrix. In addition, if we assume that the boundary conditions, (2.8), are co-normal, i.e. $\Gamma$ and $\Delta$ can be written in the form given in Theorem 6.1, and that the weight matrix $M$ commutes with $\Gamma, \Delta, \tilde{\Gamma}, \tilde{\Delta}$, then $P = U \tilde{P} U^*$. 

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Proof: From [11] we have $\Delta = U\tilde{\Delta}$ and $\Gamma = U\tilde{\Gamma}$, where $U$ is as defined in the statement of the theorem.

In order to prove $P = U\tilde{P}U^*$ we assume that $M$ commutes with $\Gamma, \Delta, \tilde{\Gamma}, \tilde{\Delta}$ and that the boundary conditions, (2.8), are co-normal, i.e. $\Gamma$ and $\Delta$ can be written in the form given in Theorem 6.1. Then the solution $Y(t)$ to $-MM'' = \lambda Y$ obeying (2.8) is given by

$$
Y(t) = \cos(\sqrt{\lambda}M^{-\frac{1}{2}}(t - 1))\Delta \hat{\delta} + \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}M^{-\frac{1}{2}}(t - 1))M^{\frac{1}{2}}\Gamma \hat{\delta}
$$

where $\hat{\delta} = \begin{pmatrix} \alpha_1 \sqrt{1 + |\mu_1|^2} \\ \vdots \\ \alpha_n \sqrt{1 + |\mu_n|^2} \\ \gamma_{n+1} \\ \vdots \\ \gamma_{2K} \end{pmatrix}$ and $\gamma_i, i = n + 1, \ldots, 2K$ are as in Theorem 6.1.

Now, since $M$ commutes with $\Gamma$ and $\Delta$, $M$ commutes with $w_k^s$, see Appendix, giving

$$
\tilde{Y} \cdot w_k = w_k^s \left[ \cos(\sqrt{\lambda}M^{-\frac{1}{2}}(t - 1))\Delta \hat{\delta} + \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}M^{-\frac{1}{2}}(t - 1))M^{\frac{1}{2}}\Gamma \hat{\delta} \right]
$$

$$
= w_k^s \Delta \cos(\sqrt{\lambda}M^{-\frac{1}{2}}(t - 1))\hat{\delta} + w_k^s \Gamma \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}M^{-\frac{1}{2}}(t - 1))M^{\frac{1}{2}}\hat{\delta}.
$$

Thus for $k = 1, \ldots, n$

$$
\tilde{Y} \cdot w_k = \frac{1}{\sqrt{1 + |\mu_k|^2}} \left( \cos(\sqrt{\lambda}m_k^{-\frac{1}{2}}(t - 1)) + m_k^{\frac{1}{2}}\mu_k \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}m_k^{-\frac{1}{2}}(t - 1)) \right)\delta_k
$$

and for $k = n + 1, \ldots, 2K$

$$
\tilde{Y} \cdot w_k = m_k^{\frac{1}{2}} \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}m_k^{-\frac{1}{2}}(t - 1))\delta_k
$$

where since $M$ is diagonal $m_k$ is the $k^{th}$ diagonal entry of $M$ and $\delta_k$ is the $k^{th}$ entry of $\hat{\delta}$.

So for $k = 1, \ldots, n$ we have that $\tilde{Y} \cdot w_k(1) = \alpha_k$ and $(\tilde{Y} \cdot w_k)'(1) = \mu_k \alpha_k$ and for $k = n + 1, \ldots, 2K$ we have $\tilde{Y} \cdot w_k(1) = 0$ and $(\tilde{Y} \cdot w_k)'(1) = \gamma_k$ meaning that we can now use lemma 5.1 with $h = \mu_k$. For $k = n + 1, \ldots, 2K$, by lemma 5.1 there exists a kernel, $k_{\infty,m_k,\delta_k}(t,y) := K^k(t,y)$ such that,

$$
V^k[f](t) := \int_t^1 K^k(t,y)f(y) \, dy,
$$

defines a Volterra map which is also a continuous linear transformation on $\mathcal{L}^2[0,1]$, and since $\tilde{Y} \cdot w_k$ is the solution of $-m_k(\tilde{Y} \cdot w_k)'' = \lambda(\tilde{Y} \cdot w_k)$ on $[0,1]$ with $\tilde{Y} \cdot w_k(1) = 0$, then

$$
Z_k := (I + V^k)(\tilde{Y} \cdot w_k)
$$

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is the solution of \(-m_kZ_k'' + q_kZ_k = \lambda Z_k\) on \([0, 1]\) with \(Z_k(1) = 0\) and \(Z_k'(1) = (\bar{Y} \cdot \bar{w}_k)'(1) = \gamma_k\), for each \(\lambda \in \mathbb{R}\).

Also for \(k = 1, \ldots, n\) there exists a kernel, \(k_{\mu_k,m_k,q_k}(t,y) := K^k(t,y)\) such that,
\[
V^k[f](t) := \int_t^1 K^k(t,y)f(y) \, dy,
\]
defines a Volterra map which is also a continuous linear transformation on \(L^2[0, 1]\), and since \(\bar{Y} \cdot \bar{w}_k\) is the solution of \(-m_k(\bar{Y} \cdot \bar{w}_k)'' = \lambda(\bar{Y} \cdot \bar{w}_k)\) on \([0, 1]\) with \(\bar{Y} \cdot \bar{w}_k'(1) = \mu_k\), then
\[
Z_k := (I + V^k)(\bar{Y} \cdot \bar{w}_k)\]
is the solution of \(-m_kZ_k'' + q_kZ_k = \lambda Z_k\) on \([0, 1]\) with \(Z_k'(1) = \alpha_k\), for each \(\lambda \in \mathbb{R}\).

Thus
\[
\bar{Z}(t) := \sum_{k=1}^{2K} w_k Z_k(t)
\]
\[
= \sum_{k=1}^{2K} \bar{Y}(t) \cdot \bar{w}_k + \sum_{k=1}^{2K} \int_t^1 K^k(t,y)\bar{Y}(y) \cdot \bar{w}_k \, dy
\]
\[
= \bar{Y}(t) + \int_t^1 \sum_{k=1}^{2K} K^k(t,y)w_k w^*_k \bar{Y}(y) \, dy
\]
\[
= (I + V_{P,M})\bar{Y}(t),
\]
where \(V_{P,M}\) is a Volterra map.

Hence if \(\bar{Y}(t)\) is the solution of \(-MY''(t) = \lambda \bar{Y}(t)\) with \(\bar{Y}(1) = \Delta\) and \(\bar{Y}'(1) = \Gamma\), then
\[
\bar{Z}(t) = \bar{Y}(t) + \int_t^1 \sum_{k=1}^{2K} K^k(t,y)w_k w^*_k \bar{Y}(y) \, dy
\]
is the solution of \(-MZ'' + P\bar{Z}(t) = \lambda \bar{Z}(t)\) with \(\bar{Z}(1) = \Delta\) and \(\bar{Z}'(1) = \Gamma\).

We note that \((I + V_{P,M})^{-1} = I + W_{P,M}\), where \(W_{P,M}\) is Volterra, see [26, p. 26]. Thus, if \(Y\) is the solution of \(-MY'' + PY = \lambda Y\) on \([0, 1]\) with \(Y(1) = \Delta\) and \(Y'(1) = \Gamma\) then
\[
Z := (I + W_{P,M})Y
\]
is the solution of \(-MZ'' = \lambda Z\) on \([0, 1]\) with \(Z(1) = \Delta\) and \(Z'(1) = \Gamma\). Hence
\[
\tilde{Z} := U^*(I + W_{P,M})Y = U^*Z
\]
is the solution of \(-U^*MU(U^*Z)'' = \lambda(U^*Z)\) with \(\bar{Z}(1) = U^*Z(1) = U^*\Delta = \tilde{\Delta}\) and \(\bar{Z}'(1) = U^*Z'(1) = U^*\Gamma = \tilde{\Gamma}\). Since \(UM = MU\) we get that \(\bar{Z}\) is the solution of \(-MZ'' = \lambda\bar{Z}\) with \(\bar{Z}(1) = \Delta\) and \(\bar{Z}'(1) = \Gamma\).

Let
\[
\bar{Y} := (I + V_{\bar{P},M})\bar{Z} = (I + V_{\bar{P},M})U^*(I + W_{P,M})Y,
\]
then \(\bar{Y}\) is the solution of \(-MY'' + \bar{P}Y = \lambda Y\) with \(\bar{Y}(1) = \Delta\) and \(\bar{Y}'(1) = \Gamma\).

Let \(HY := (I + V_{\bar{P},M})U^*(I + W_{P,M})Y\) for \(Y \in \mathcal{L}^2[0,1]\). If \(Y\) is any solution of \(-MY'' + PY = \lambda Y\), then \(\bar{Y} := HY\) is the solution of \(-MY'' + \bar{P}Y = \lambda Y\) with \(\bar{Y}(1) = U^*Y(1)\) and \(\bar{Y}'(1) = U^*Y'(1)\). In particular \(HW_3 = \tilde{W}_3\) and \(HW_2 = \tilde{W}_2\), and hence, by Lemma 5.2, \(H\) is unitary.

Let, \(N := UH\), then
\[
N := U(I + V_{\bar{P},M})U^*(I + W_{P,M}) = I + W_{P,M} + UV_{\bar{P},M}U^* + UV_{\bar{P},M}U^*W_{P,M} = I + J,
\]
where \(J\) is Volterra. We now show \(N\) is unitary. For \(g, f \in \mathcal{L}^2[0,1]\), we have
\[
< g, N^* f > = < Ng, f >,
\]
which implies
\[
\int_0^1 g^T M^{-1} \overline{N^* f} \, dt = \int_0^1 (Ng)^T M^{-1} f \, dt.
\]
But \(N = UH\) so
\[
\int_0^1 g^T M^{-1} (UH)^* f \, dt = \int_0^1 (Ug)^T M^{-1} \bar{f} \, dt = \int_0^1 (Hg)^T U^T M^{-1} \bar{f} \, dt = \int_0^1 (Hg)^T \overline{U^*M^{-1} f} \, dt.
\]
Since \(MU = UM\) we have \(M^{-1}U^* = U^*M^{-1}\). Hence,
\[
\int_0^1 g^T M^{-1} (UH)^* f \, dt = \int_0^1 (Hg)^T U^T M^{-1} \overline{f} \, dt = \int_0^1 g^T M^{-1} H^* (U^* f) \, dt.
\]
Thus \((UH)^* = H^* U^* = H^{-1} U^*\), where the latter equality is due to \(H\) being unitary. Hence \((UH)(UH)^* = I\) and \((UH)^*(UH) = I\). Therefore \(N\) is unitary.

But \(N = I + J\) where \(J\) is Volterra and \(N\) is unitary, so by [6, p.93], we have \(J = 0\) and \(N = I\), giving \(U^* = (I + V_{\bar{P},M})U^*(I + W_{P,M})\). Therefore \(\tilde{W}_3 = U^* W_3\) and \(\tilde{W}_2 = U^* W_2\), and
\[
-M(U^*W_j)'' + \bar{P}U^*W_j = \lambda U^*W_j,
\]
for \(j = 2, 3\).
Premultiplying equation (5.2) by $U$ and noting that $MU = UM$, gives

$$-MW_j'' + U\tilde{P}U^*W_j = \lambda W_j,$$

for $j = 2, 3$, but we also have that

$$-MW_j'' + PW_j = \lambda W_j,$$

for $j = 2, 3$. So

$$(U\tilde{P}U^* - P)W_j = 0,$$

for $j = 2, 3$, and for all $d \in \mathbb{R}^{4K}$, $(U\tilde{P}(t)U^* - P(t))[W_2(t), W_3(t)]d = 0$ for all $t \in [0, 1]$. But the column space of $[W_2, W_3]$ spans the solution space of $-MY'' + PY = \lambda Y$, so for each $t_0 \in [0, 1]$ and $c \in \mathbb{R}^{2K}$ there exists $d \in \mathbb{R}^{4K}$ such that $c = [W_2(t_0), W_3(t_0)]d$ giving $(U\tilde{P}(t_0)U^* - P(t_0))c = 0$ for each $c \in \mathbb{R}^{2K}$. Thus $P = U\tilde{P}U^*$.

**Remark** In fact, for the above result, we only require $M_n = \tilde{M}_n$ and not that the entire $M$-matrices are equal.

The following two corollaries are immediate consequences of the above theorem.

**Corollary 5.4** If all the edges of the graph $G$ have the same length, $l$, and the systems problems $(\Gamma^*, \Delta^*, P)$ and $(\tilde{\Gamma}^*, \tilde{\Delta}^*, \tilde{P})$ have the same $M$-matrix, then $\Delta = U\tilde{\Delta}$, $\Gamma = U\tilde{\Gamma}$ and $P = U\tilde{P}U^*$ where $U = \Gamma\tilde{\Gamma}^* + \Delta\tilde{\Delta}^*$ is a unitary matrix.

**Proof:** Since all the edges of the graph $G$ have the same length $l$ we have that $M = \frac{1}{l^2}I$ which commutes with $\Gamma, \Delta, \tilde{\Gamma}, \tilde{\Delta}$. The result now follows from Theorem 5.3.

**Corollary 5.5** If the problems $(\Gamma^*, \Delta^*, P)$ and $(\Gamma^*, \Delta^*, \tilde{P})$ have the same $M$-matrix and $M$ commutes with $\Gamma, \Delta$ then $P = \tilde{P}$.

**Proof:** In the notation of Theorem 5.3, $\tilde{\Gamma} = \Gamma$, $\tilde{\Delta} = \Delta$ and hence $U = I$. The result follows immediately from Theorem 5.3.

Note that given a set of eigenvalues $\lambda_n$ and the terminal value, $\Delta^*F_{n,j}(1) + \Gamma^*F_{n,j}'(1) = \Delta^*W_3(\lambda_n, 1)c_{n,j} + \Gamma^*W_3'(\lambda_n, 1)c_{n,j} = c_{n,j}$, $M_n$ is uniquely determined and the above corollaries apply.

**Remark** The above note is actually a more appealing result since it means that from the eigenvalues and the data at the nodes of the given graph, i.e. the terminal conditions, we can recover the boundary conditions and the potential. It does not rely on the superficial nodes inserted into each edge only on the original given nodes.
The definition of co-normal boundary conditions on a graph is given in [10]. An immediate consequence of this is that for the system formulation (2.6)-(2.8) the boundary conditions (2.8), at \( x = 1 \), are co-normal if and only if \( \Gamma \) and \( \Delta \) are such that, when

\[
S = \left\{ \begin{pmatrix} u \\ u' \end{pmatrix} \in \mathbb{C}^{2K} \bigoplus \mathbb{C}^{2K} \mid \Gamma^* u - \Delta^* u' = 0 \right\},
\]

is such that there exists a subspace \( N \), of dimension \( n \), of \( \mathbb{C}^{2K} \) so that \( \begin{pmatrix} u \\ 0 \end{pmatrix} \in S \) for all \( u \in N \) and there exists a real diagonal matrix \( D =: \text{diag}\{d_1, \ldots, d_{2K}\} \) such that \( \begin{pmatrix} u \\ u' \end{pmatrix} \in S \) if and only if \( u \in N \) and \( (Du - u') \cdot v = 0 \) for all \( v \in N \).

We remark that co-normal boundary conditions on a graph correspond in nature to co-normal (non-oblique) boundary conditions for elliptic partial differential operators. Most physically interesting boundary conditions on graphs fall into the co-normal category. In particular, ‘Kirchhoff’, Dirichlet, Neumann and periodic boundary conditions are all co-normal, but this class does not include all self-adjoint boundary-value problems on graphs. For example consider a single loop, i.e. the interval \([0, 1]\) where the boundary conditions at 0 and at 1 are connected as follows \( y(0) = y'(1) \) and \( y(1) = -y'(0) \). These boundary conditions give a self-adjoint boundary-value problem with non co-normal boundary conditions.

**Theorem 6.1** Suppose that the boundary conditions, (2.8), are co-normal and that \( S, N, D \) are as given above. Then there exists an orthonormal basis \( w_1, \ldots, w_{2K} \) for \( \mathbb{C}^{2K} \) and real numbers \( \mu_1, \ldots, \mu_n \) such that, without loss of generality, \( \Delta \) and \( \Gamma \) may be written as

\[
\Delta = \begin{bmatrix} \frac{w_1}{\sqrt{1 + |\mu_1|^2}} \ldots, \frac{w_n}{\sqrt{1 + |\mu_n|^2}}, 0, \ldots, 0 \end{bmatrix},
\]

and

\[
\Gamma = \begin{bmatrix} \frac{\mu_1 w_1}{\sqrt{1 + |\mu_1|^2}} \ldots, \frac{\mu_n w_n}{\sqrt{1 + |\mu_n|^2}}, w_{n+1}, \ldots, w_{2K} \end{bmatrix}.
\]

**Proof:** Let \( v_1, \ldots, v_n \) be an orthonormal basis for \( N \) and let \( v_{n+1}, \ldots, v_{n+m} \), where \( n + m = 2K \), be the extension of this basis for \( N \) to an orthonormal basis for \( \mathbb{C}^{2K} \). Now \( \begin{pmatrix} u \\ u' \end{pmatrix} \in S \) if and only if

\[
u = \sum_{i=1}^{n} \alpha_i v_i \quad \text{and} \quad Du - u' = - \sum_{j=1}^{2K-n-m} \beta_j v_{j+n},
\]
for $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $\beta_1, \ldots, \beta_m \in \mathbb{C}$.

I.e.

$$u' = Du + \sum_{j=1}^{m} \beta_j v_{j+n} = \sum_{i=1}^{n} \alpha_i Dv_i + \sum_{j=1}^{m} \beta_j v_{j+n}. $$

In particular we need

$$u' \cdot v_k = \sum_{i=1}^{n} \alpha_i (Dv_i) \cdot v_k + \sum_{j=1}^{m} \beta_j v_{j+n} \cdot v_k, \quad k = 1, \ldots, 2K. $$

So for $k = 1, \ldots, n$, we get

$$u' \cdot v_k = \sum_{i=1}^{n} \alpha_i (Dv_i) \cdot v_k, $$

and for $k = n+1, \ldots, 2K$ we get

$$u' \cdot v_k = \sum_{i=1}^{n} \alpha_i (Dv_i) \cdot v_k + \beta_k. $$

Since $\beta_k$ are arbitrary we can set

$$\sum_{i=1}^{n} \alpha_i (Dv_i) \cdot v_k + \beta_k := \gamma_k $$

giving $\gamma_k$ an arbitrary element of $\mathbb{C}$ and

$$u' \cdot v_k = \begin{cases} \sum_{i=1}^{n} \alpha_i (Dv_i) \cdot v_k, & k = 1, \ldots, n \\ \gamma_k, & k = n+1, \ldots, 2K. \end{cases} $$

Thus

$$u' = \sum_{i,k=1}^{n} v_k ((Dv_i) \cdot v_k) \alpha_i + \sum_{k=1}^{m} \gamma_{k+n} v_{k+n} $$

$$= \sum_{i,k=1}^{n} v_k d_{k,i} \alpha_i + \sum_{k=1}^{m} \gamma_{k+n} v_{k+n} $$

where $d_{k,i} = (Dv_i) \cdot v_k$. Since $D$ is real and diagonal, $[d_{k,i}]$ is Hermitian giving that $[d_{k,i}]$ is diagonalizable and the eigenvalues are semisimple. Hence $[d_{k,i}]$ has real eigenvalues $\mu_1, \ldots, \mu_n$, say, with orthonormal eigenvectors say $z_1, \ldots, z_n \in \mathbb{C}^n$. I.e. $[d_{k,i}] z_j = \mu_j z_j = z_j \mu_j$ giving $[d_{k,i}] [z_1, \ldots, z_n] = [z_1, \ldots, z_n] [\mu_i]$ where $[\mu_i] = \text{diag} [\mu_1, \ldots, \mu_n]$. 

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Therefore
\[
\sum_{i,k=1}^{n} v_k d_{k,i} \alpha_i = \sum_{i,k=1}^{n} v_k d_{k,i} (u \cdot v_i)
\]
\[
= [v_1, \ldots, v_n] [d_{k,i}] \begin{bmatrix} v_1^* u \\ \vdots \\ v_n^* u \end{bmatrix}
\]
\[
= [v_1, \ldots, v_n] [z_1, \ldots, z_n] [\mu_1, \ldots, \mu_n]^* \begin{bmatrix} v_1^* u \\ \vdots \\ v_n^* u \end{bmatrix}
\]
\[
= [w_1, \ldots, w_n] [\mu_1, \ldots, \mu_n]^* u.
\]

Here, since \([z_1, \ldots, z_n]\) is an \(n \times n\) unitary matrix we have that \([w_1, \ldots, w_n]\) as given by \([w_1, \ldots, w_n] = [z_1, \ldots, z_n][\mu_1, \ldots, \mu_n]^*\) is an orthonormal basis for \(N\).

Setting \(w_{n+k} = v_{n+k}\) for \(k = 1, \ldots, m\) we have that
\[
u' = \sum_{k=1}^{n} w_k \mu_k (u \cdot w_k) + \sum_{k=1}^{m} \gamma_{k+n} w_{k+n}.
\]

The importance of this mapping is that
\[
u' \cdot w_k = \mu_k (u \cdot w_k)
\]
for all \(k = 1, \ldots, n\) and
\[
u' \cdot w_k = \gamma_k
\]
for all \(k = n + 1, \ldots, 2K\) where
\[
u \cdot w_k = \begin{cases} u \cdot w_k, & k = 1, \ldots, n \\ 0, & k = n + 1, \ldots, 2K \end{cases}
\]

Hence \(\Delta\) and \(\Gamma\) may be written as
\[
\Delta = \begin{bmatrix} \frac{w_1}{\sqrt{1 + |\mu_1|^2}}, \ldots, \frac{w_n}{\sqrt{1 + |\mu_n|^2}}, 0, \ldots, 0 \end{bmatrix}
\]
and
\[
\Gamma = \begin{bmatrix} \frac{\mu_1 w_1}{\sqrt{1 + |\mu_1|^2}}, \ldots, \frac{\mu_n w_n}{\sqrt{1 + |\mu_n|^2}}, w_{n+1}, \ldots, w_{2K} \end{bmatrix}.
\]

It should be noted that the identities \(\Delta^* \Delta + \Gamma^* \Gamma = I\) and \(\Gamma^* \Delta = \Delta^* \Gamma\) still hold.

Similarly we may assume, without loss of generality, that \(\tilde{\Gamma}\) and \(\tilde{\Delta}\) can be written in the form of Theorem 6.1.
References


[3] P.A. Binding, P.J. Browne, B.A. Watson, Recovery of the m-function from spectral data for generalised Sturm-Liouville problems,


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